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EXTENSION AND CONVERGENCE THEOREMS OF PSEUDOHOLOMORPHIC MAPS

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Abstract

First, we generalize the big Picard theorem for pseudoholomorphic maps. Next, we prove Noguchi-type extension-convergence theorems for pseudoholomorphic maps. Finally, we show that the J-automorphism group of an almost complex submanifold hyperbolically embedded in a compact, almost complex manifold of real dimension 4 is finite when it is the complement of an immersed pseudoholomorphic curve.

Introduction

The classical big Picard theorem states that if $f\colon \Delta^* \to \mathbb{P}_1(\mathbb{C})$ is holomorphic from the punctured unit disk of the complex plane to the Riemann sphere $\mathbb{P}_1(\mathbb{C})$ and $\mathbb{P}_1(\mathbb{C}) \setminus f(\Delta^*)$ has more than two elements, then f has a holomorphic extension $\tilde{f}\colon \Delta \to \mathbb{P}_1(\mathbb{C})$. This theorem has been extended to higher dimensional settings by Kiernan [11], Kwack [20] and Kobayashi [14] by introducing the notion of hyperbolic embeddedness. With modifications of the Kiernan's proof [11], Adachi [1] (cf. Theorem 2.2) and later Joseph-Kwack [9] (cf. Theorem 1) proved an extension theorem of holomorphic mappings.

The first aim of this paper is to give a generalization of the Adachi theorem. We emphasize that his proof is based on the Riemann extension theorem and winding numbers arguments. So in order to apply it to the almost complex case, we need substantial modifications.

Theorem 1. *Let (M, J) be a relatively compact almost complex submanifold in an almost complex manifold (N, J) and let $f_k\colon \Delta^* \to (M, J)$ be a sequence of pseudo-holomorphic curves. Let (z_k) and (w_k) be sequences in Δ^* converging to 0 such that the sequence $(f_k(w_k))$ converges to $q \notin S_M^J(N)$. Then the sequence $(f_k(z_k))$ converges to q.*

Here $S_M^J(N)$ denotes the set of degeneracy points of d_M^J, see Section 2.

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Next, we give a generalization of a result proved by Noguchi in the complex case, (cf. [25] Lemma 2.4 p. 21)

Theorem 2. *Let (M, J) be a relatively compact almost complex submanifold hyperbolically embedded in an almost complex manifold (N, J). Then $\mathcal{O}_J(\Delta^*, M)$ is relatively compact in $\mathcal{O}_J(\Delta, N)$.*

We investigate the extension of pseudoholomorphic maps into hyperbolically embedded almost complex manifolds.

Theorem 3. *Let C be a smooth pseudoholomorphic curve in an almost complex manifold (S, J') of real dimension 4 and let (M, J) be a relatively compact almost complex submanifold hyperbolically embedded in an almost complex manifold (N, J). Then every pseudoholomorphic map $f\colon (S\setminus C, J') \to (M, J)$ extends to a (J', J)-holomorphic map from S to N.*

Noguchi [25] proved a remarkable theorem about the preservation of uniform convergence on compact subsets by holomorphic extensions which we recall here: let X be a relatively compact hyperbolically embedded complex submanifold of a complex manifold Y. Let M be a complex manifold and A a complex hypersurface of M with only normal crossings. If $(f_j\colon M \setminus A \to X)_j$ is a sequence of holomorphic mappings which converges uniformly on compact subsets of $M \setminus A$ to a holomorphic mapping $f\colon M \setminus A \to X$, then the sequence $(\tilde{f}_j)$ converges uniformly on compact subsets of M to $\tilde{f}$, where $\tilde{f}_j\colon M \to Y$ and $\tilde{f}\colon M \to Y$ are the unique holomorphic extensions of f_j and f over M.

The second aim of this paper is to generalize Noguchi theorem for almost complex manifolds and we give other variants.

The following extension-convergence theorem is a generalization to the almost complex case of Joseph-Kwack theorem (see [9] Theorem 1 p. 364).

Theorem 4. *Let (M, J) be a relatively compact almost complex submanifold in an almost complex manifold (N, J). Let*

$$f_n\colon \Delta^* \to (M, J)$$

and

$$f\colon \Delta^* \to (M, J)$$

be a sequence of pseudoholomorphic curves and a pseudoholomorphic curve respectively. We suppose that there exists a sequence (z_n) in Δ^* converging to 0 such that the sequence $(f_n(z_n))$ converges to $p \notin S_M^J(N)$. Then f and f_n extends to Δ and if the sequence (f_n) converges to f uniformly on compact subsets of Δ^*, then the sequence $(\tilde{f}_n)$

converges to $\tilde{f}$ *uniformly on compact subsets of* Δ, *where* $(\tilde{f}_n)$ *and* $\tilde{f}$ *are the extension to* Δ *of* f_n *and* f *respectively.*

In higher dimensional settings, we prove a variant of Noguchi-type extension convergence theorems for pseudo-holomorphic maps defined on $S \setminus C$ valued into hyperbolically embedded submanifold, where C is smooth pseudoholomorphic curve in an almost complex manifold S of real dimension 4.

Theorem 5. *Let* C *be a smooth pseudoholomorphic curve in an almost complex manifold* (S, J') *of real dimension* 4 *and let* (M, J) *be a relatively compact almost complex submanifold hyperbolically embedded in an almost complex manifold* (N, J). *Let*

$$f_n : (S \setminus C, J') \to (M, J)$$

and

$$f : (S \setminus C, J') \to (M, J)$$

be a sequence of (J', J)*-holomorphic maps and a* (J', J)*-holomorphic map respectively.*

If the sequence (f_n) *converges to* f *uniformly on compact subsets of* $S \setminus C$, *then the sequence* $(\tilde{f}_n)$ *converges to* $\tilde{f}$ *uniformly on compact subsets of* S, *where* $\tilde{f}_n$ *and* $\tilde{f}$ *are the* (J', J)*-holomorphic extension to* S *of* f_n *and* f *respectively.*

When M is compact hyperbolic, we can strengthen the last theorem in the following form.

Theorem 6. *Let* A *be a thin subset in an almost complex manifold* (X, J'), (M, J) *be a compact hyperbolic almost complex manifold and* (J_n) *be a sequence of almost complex structures on* M *converging to* J *for the* $\mathcal{C}^\infty$*-topology.*

Let

$$f_n : (X \setminus A, J') \to (M, J_n)$$

and

$$f : (X \setminus A, J') \to (M, J)$$

be a sequence of pseudo-holomorphic maps and a pseudo-holomorphic map respectively.

If the sequence (f_n) *converges to* f *uniformly on compact subsets of* $X \setminus A$, *then the sequence* $(\tilde{f}_n)$ *converges to* $\tilde{f}$ *uniformly on compact subsets of* X, *where* $\tilde{f}_n : (X, J') \to (M, J_n)$ *and* $\tilde{f} : (X, J') \to (M, J)$ *are the extension to* X *of* f_n *and* f *respectively.*

For the definition of a thin subset of an almost complex manifold see Section 2.3.

After defining the notion of hyperconvex domain and the almost complex case, we give another variant of the Noguchi extension-convergence theorem for J-holomorphic curves. This generalizes the result of D. Thai in the complex case [30]. His proof is based on Kwack's extension theorem and the maximum principle.

Theorem 7. *Let (M, J) be a relatively compact hyperbolic almost complex submanifold in an almost complex manifold (N, J). Assume that there is a neighborhood U of ∂M, the boundary of M in N such that $U \cap M$ is hyperconvex. Then, each pseudoholomorphic curve $f \colon \Delta^* \to M$ extends to a pseudoholomorphic curve $\tilde{f} \colon \Delta \to M$.*

Moreover, if $(f_n \colon \Delta^* \to (M, J))$ is a sequence of pseudoholomorphic curves that converges uniformly on compact subsets of Δ^* to a pseudoholomorphic curve $f \colon \Delta^* \to (M, J)$, then the sequence $(\tilde{f}_n)$ converges uniformly on compact subsets of Δ to $\tilde{f}$, where $\tilde{f}_n$ and $\tilde{f}$ are the extension to Δ of f_n and f respectively.

In the last section, we prove the following

Theorem 8. *Let (C, J) be a pseudoholomorphic curve in a compact almost complex manifold (S, J) of dimension 4, such that $S \setminus C$ is hyperbolically embedded in S. Then the automorphism group $\mathrm{Aut}_J(S \setminus C)$ of $S \setminus C$ is finite.*

This is a "non-compact" version of the theorems of Kruglikov-Overholt [19] and Kobayashi [17] which asserts that the automorphism group of a compact hyperbolic almost complex manifold is finite. We recall that in the complex case, this is due to Kobayashi [16]. Miyano-Noguchi [23] showed that the automorphism group of $X \setminus D$ is finite where D is a normal crossing divisor in X and $X \setminus D$ is hyperbolically embedded in X. We have considered the four dimensional case since a hypersurface in an almost complex manifold does not exists even locally in general.

1. Preliminaries

1.1. Almost complex manifolds. 1. An almost complex manifold (M, J) is a smooth real manifold equipped with an almost complex structure J. We remind that an almost complex structure J is a smooth tensor field of type (1, 1), such that $J^2 = -\mathrm{Id}$. An almost complex structure is called an (integrable) complex structure if it is the almost complex structure defined by an atlas on M of complex coordinate charts that overlap biholomorphically.

Given two almost complex manifolds (M, J) and (M', J') and a smooth map $f \colon M' \to M$ is called (J', J)-holomorphic if its differential $df \colon TM' \to TM$ verifies

$$df \circ J' = J \circ df$$

on TM. We denote by $\mathcal{O}_{(J',J)}(M', M)$ the set of (J', J)-holomorphic maps from M' to M.

2. For every $r > 0$, we set $\Delta_r = \{z \in \mathbb{C},\ |z| < r\}$ and $\Delta = \Delta_1$ is the unit disc in $\mathbb{C}$. If $(M', J') = (\Sigma, J_0)$ where J_0 denotes the standard complex structure on a Riemann surface Σ, a (J_0, J)-holomorphic map is called a *J-holomorphic curve* or a *pseudo-holomorphic curve* and we denote by $\mathcal{O}_J(\Sigma, M)$ the set of J-holomorphic curves in M.

3. By a length function on M, we mean a continuous function $G\colon TM \to [0, +\infty[$ satisfying

(i) $G(v) = 0$, if and only, if $v = 0$,

(ii) for all real numbers c, we have $G(cv) = |c|G(v)$.

Let d_G be the distance function generated on M by G, see [21]. For simplicity, we denote $|v|_G$ instead of $G(v)$ for $v \in TM$.

4. We say that an upper semicontinuous function u on (M, J) is *plurisubharmonic* if its composition with any J-holomorphic curve is subharmonic. For a $\mathcal{C}^2$ function this is equivalent to the positive semi-definiteness of the Levi form:

$$L_u^J(p, \xi) \geq 0 \quad \text{for any} \quad p \in M \quad \text{and} \quad \xi \in T_p(M).$$

The value of the Levi form of u at a point $p \in M$ and a vector $\xi \in T_p(M)$ is defined by

$$L_u^J(p, \xi) := -d(J^* du)(X, JX),$$

where X is an arbitrary smooth vector field in a neighborhood of p satisfying $X(p) = \xi$.

We shall remark that if $z\colon \Delta \to M$ is a J-holomorphic curve satisfying $z(0) = p$ and $dz(0)(\partial/\partial x) = \xi$, then $L_u^J(p, \xi) = \Delta(u \circ z)(0)$, see [8].

We say that a $\mathcal{C}^2$ real valued function u on M is *strictly J-plurisubharmonic* on M if L_u^J is positive definite on TM.

Let M be a relatively compact domain in an almost complex manifold (N, J). Recall that a continuous proper map $u\colon M \to \mathbb{R}$ is called an *exhaustion function* for a domain M.

DEFINITION 1.1. 1. A domain M in an almost complex manifold (N, J) is a Stein domain if there exists a strictly plurisubharmonic exhaustion function on M.
2. An almost complex manifold (M, J) is said to be hyperconvex if M is Stein and there exists a continuous plurisubharmonic function $u\colon M \to]-\infty, 0[$ such that $M_c = \{x \in M;\ u(x) \leq c\}$ is compact for every $c < 0$.

1.2. Kobayashi hyperbolicity of almost complex manifolds. Let (M, J) be an almost complex manifold. A Kobayashi chain joining two points $p, q \in M$ is a sequence of pseudoholomorphic curves $(f_k\colon \Delta \to (M, J))_{1 \leq k \leq m}$ and points $z_k, w_k \in \Delta$

such that $f_1(z_1) = p$, $f_k(w_k) = f_{k+1}(z_{k+1})$ and $f_m(w_m) = q$. The Kobayashi pseudo-distance of (M, J) from p to q is defined by

$$d_M^J(p, q) = \inf \sum_{k=1}^{m} d_\Delta(z_k, w_k),$$

where the infimum is taken over all Kobayashi chains joining p to q and d_Δ denotes the Poincaré distance of Δ. Recall that for every two sufficiently close points p and q in M, there exists a J-holomorphic curve $u\colon \Delta \to M$ such that $u(0) = p$ and $u(z_0) = q$, where $z_0 \in \Delta$.

For every $p \in M$, there is a neighborhood $\mathcal{V}$ of 0 in T_pM such that for every $\xi \in \mathcal{V}$ there exist a constant $c > 0$ and $f : \Delta \to (M, J)$ a pseudoholomorphic curve satisfying $f(0) = p$ and $f'(0) = c.\xi$. This allows one to define the Kobayashi-Royden infinitesimal pseudometric K_M^J.

$$K_M^J(p, \xi) = \inf \left\{ \frac{1}{r};\ f : \Delta_r \to M,\ J\text{-holomorphic};\ f(0) = p,\ f'(0) = \xi \right\}.$$

We deduce the non increasing property which can be stated as follows: Let $f : (M', J') \to (M, J)$ be a (J', J)-holomorphic map. Then

$$f^* K_M^J \le K_{M'}^{J'}.$$

Kruglikov [18] extended Royden's results [27] and proved that K_M^J is upper semi-continuous on the tangent bundle TM of M and that the integrated form of the Kobayashi-Royden metric K_M^J coincides with the pseudo-distance d_M^J of Kobayashi.

We say that (M, J) is *hyperbolic* if d_M^J is a distance. A hyperbolic almost complex manifold (M, J) is said to be *complete* if it is Cauchy complete with respect to d_M^J. An open subset U in an almost complex manifold (M, J) is called *locally complete hyperbolic* if every p of the closure $\overline{U}$ has a neighborhood V_p in M such that $V_p \cap U$ is complete hyperbolic. Recall that the complement of a pseudoholomorphic curve in an almost complex surface is locally complete hyperbolic, see [6].

Let (M, J) be an almost complex submanifold of an almost complex manifold (N, J). Then M is said to be *hyperbolically embedded* in N, if for every pair (p, q) of different points in $\overline{M}$, there exist neighborhoods U and V of p and q in N such that $d_M^J(M \cap U, M \cap V) > 0$.

A compact almost complex manifold (M, J) is hyperbolic if and only if, every pseudoholomorphic curve $f : \mathbb{C} \to M$ is constant, see [19].

2. Generalization of the big Picard theorem

The big Picard theorem states that any holomorphic map f from the punctured disk Δ^* into $\mathbb{C} \setminus \{0, 1\}$ can be extended to a holomorphic map $f : \Delta \to \mathbb{P}_1(\mathbb{C})$. The aim of this section is to study the extension of pseudoholomorphic maps in different situations.

2.1. Degeneracy locus of the Kobayashi pseudodistance. Let (N, J) be an almost complex manifold equipped with a length function G and let (M, J) be a relatively compact almost complex submanifold of (N, J). As in [2], we extended d_M^J to the closure $\overline{M}$ of M in N as follows: for $p, q \in \overline{M}$, we define

$$d_{\overline{M}}^J(p, q) = \liminf_{p' \to p, q' \to q} d_M^J(p', q'), \quad p', q' \in M.$$

DEFINITION 2.1. We call $p \in \overline{M}$ a degeneracy point of $d_{\overline{M}}^J$ if there exists a point $q \in \overline{M} \setminus \{p\}$ such that $d_{\overline{M}}^J(p, q) = 0$. We denote by $S_M^J(N)$ the set of the degeneracy points of $d_{\overline{M}}^J$.

EXAMPLE 2.2. Let $(N, J) = (\mathbb{P}_1(\mathbb{C}), J_0)$ be the Riemann sphere equipped with its standard complex structure J_0 and $M = \mathbb{C} \setminus \{0\}$. Since $d_M^{J_0} = 0$ (see Kobayashi [16] Example 3.1.21 p. 56), then

$$S_{\mathbb{C}\setminus\{0\}}^{J_0}(\mathbb{P}_1(\mathbb{C})) = \mathbb{P}_1(\mathbb{C}),$$

that is all points of $\mathbb{P}_1(\mathbb{C})$ are degeneracy points of $d_{\overline{M}}^{J_0}$.

DEFINITION 2.3. Let (N, J) be an almost complex manifold equipped with a length function G and let (M, J) be a submanifold of N. A point $p \in \overline{M}$ is called a J-hyperbolic point for M if there exists a neighborhood U of p in N and a positive constant c such that $K_M^J \geq c \,.\, G$ on $U \cap M$.

REMARK 2.4. An easy adaptation of the infinitesimal criterion of hyperbolicity due to Royden [27] shows that, (M, J) is hyperbolic, if and only if, every point p in M is a J-hyperbolic point for M.

We prove the following

Proposition 2.5. *Let (N, J) be an almost complex manifold and let (M, J) be an almost complex submanifold of (N, J). For a point p in $\overline{M}$, the following are equivalent*:

(i) $p \notin S_M^J(N)$.

(ii) *p is a J-hyperbolic point for M.*

For the proof, we will use the following lemma (cf. [28] Proposition 2.3.6, p. 171).

Lemma 2.6. *Let D be a domain in $\mathbb{C}^n$. There is a positive constant δ_0 such that for every almost complex structure J in a neighborhood of $\overline{D}$ satisfying $\| J - J_0 \|_{\mathcal{C}^2(\overline{D})} \leq \delta_0$, we have*

$$\| f \|_{\mathcal{C}^1(\Delta_r)} \leq c \| f \|_{\mathcal{C}^0(\Delta)},$$

for every $f \in \mathcal{O}_J(\Delta, D)$ and for every $0 < r < 1$, where c is a positive constant depending only on r and δ_0. J_0 denotes the standard complex structure on $\mathbb{C}^n$.

Proof of Proposition 2.5. (i) $\Rightarrow$ (ii) Assume that p is not a J-hyperbolic point for M. Then for every $n \geq 1$, there exist $p_n \in M$ and $\xi_n \in T_{p_n} M$ such that the sequence (p_n) converges to p, $|\xi_n| = 1$ and $K_M^J(\xi_n) \to 0$. Hence, there exists a sequence (f_n) in $\mathcal{O}_J(\Delta, M)$ such that $\lim f_n(0) = p$, but $\lim |f_n'(0)| = \infty$.

Let W be a relatively compact neighborhood sufficiently small in some local chart around p. If there is a number $r \in]0, 1[$ such that $f_n(\Delta_r) \subset W$, we deduce by Lemma 2.6 that there is a positive constant c, verifying $|f_n'(0)| \leq c \| f_n \|_{\mathcal{C}^0(\Delta_r)}$ and this contradicts $|f_n'(0)| \to +\infty$. Hence, for each positive integer k there are $z_k \in \Delta$ and an integer n_k such that $|z_k| < 1/k$ and $f_{n_k}(z_k) \in \partial W$. By taking a subsequence, we may assume that $f_{n_k}(z_k) \to q \in \partial W$. Then

$$d_{\overline{M}}^J(p, q) = \lim_{k \to \infty} d_M^J(f_{n_k}(0), f_{n_k}(z_k)) \leq \lim_{k \to \infty} d_\Delta(0, z_k) = 0,$$

so p is a degeneracy point of $d_{\overline{M}}^J$.

(ii) $\Rightarrow$ (i) Assume that p is a degeneracy point of $d_{\overline{M}}^J$. Then there exists a point $q \in \overline{M} \setminus \{p\}$ such that $d_{\overline{M}}^J(p, q) = 0$. By hypothesis, there exists a neighborhood U of p such that $q \notin \overline{U}$ and $K_M^J \geq cG$ on $U \cap M$ where c is a positive constant and G is a length function on N. Take neighborhoods V, W of p, q in N respectively, such that $\overline{V} \Subset U$ and $\overline{W} \cap \overline{U} = \emptyset$. Let $r \in V \cap M$ and $s \in W \cap M$ be arbitrary points. Let $\gamma(t)$ be any piecewise smooth curve on M such that $\gamma(0) = r$ and $\gamma(1) = s$. Then

$$d_M^J(r, s) = \inf_\gamma \int_0^1 K_M^J(\gamma(t), \gamma'(t))\, dt \geq \inf_\gamma c \int_{t \in E} |\gamma'(t)|_G \, dt \geq c \,.\, \mathrm{dist}(\partial U, \partial V),$$

where $E = \{t \in [0, 1];\ \gamma(t) \in U\}$. This implies that $d_{\overline{M}}(p, q) \geq c \,.\, \mathrm{dist}(\partial U, \partial V) > 0$. This is a contradiction. $\square$

REMARK 2.7. From the proof above, we can immediately deduce that $p \notin S_M^J(N)$, if and only if, p satisfies the following condition: for every neighborhood W of p there exists a positive constant R such that

$$\sup_{f \in \mathcal{O}_J(\Delta, M)} \{ |f'(0)|,\ f(0) \in W \} \leq R.$$

From Proposition 2.5, we have the following corollaries.

Corollary 2.8. *Let (N, J) be an almost complex manifold and let (M, J) be a submanifold of N. Then (M, J) is hyperbolically embedded in (N, J), if and only if, $S_M^J(N) = \emptyset$.*

Corollary 2.9. *$S_M^J(N)$ is a closed subset in N.*

Proof. Let (p_n) be a sequence in $S_M^J(N)$ converging to some point $p \in \overline{M}$. Then by Proposition 2.5, there exist a relatively compact neighborhood W of p and $q_n \in \partial W \cap \overline{M}$ such that $d_{\overline{M}}^J(p_n, q_n) = 0$. By taking a subsequence we may assume that $q_n \to q \in \partial W$ then $d_{\overline{M}}^J(p, q) = 0$. □

2.2. Extension of *J*-holomorphic curves. The following is a generalization of the Adachi theorem [1].

Theorem 2.10. *Let (M, J) be a relatively compact almost complex submanifold in an almost complex manifold (N, J) and let $f_k \colon \Delta^* \to (M, J)$ be a sequence of pseudoholomorphic curves. Let (z_k) and (w_k) be sequences in Δ^* converging to 0 and such that the sequence $(f_k(w_k))$ converges to $q \notin S_M^J(N)$. Then the sequence $(f_k(z_k))$ converges to q.*

REMARK 2.11. The condition that $q \notin S_M^J(N)$ is essential in Theorem 2.10 as shown by this example: for each $q \in \mathbb{C} \setminus \{0\}$ and positive integer k, define the sequence of holomorphic curves $f_k \colon \Delta^* \to \mathbb{C} \setminus \{0\}$ by $f_k(z) = q/(kz)$. We have $f_k(1/k) = q$ and $f_k(1/(2k)) = 2q$.

For the proof, we will need the following lemma known as Gromov's monotonicity lemma, see [24] Lemma 4.2.1 p. 223. In the complex case, we have a similar estimation for analytic subsets, see Stolzenberg [29]. We should note that the condition on the boundary can be removed for smooth holomorphic curves, see [26] p. 30.

Lemma 2.12. *Let (M, J) be a compact almost complex manifold equipped with a length function G. Denote by $B(x, \varepsilon)$ the ball with radius ε centered at x in M. There are positive constants ε_0 and c such that for every $\varepsilon \le \varepsilon_0$ and every pseudoholomorphic curve S, it holds that*

$$\mathrm{Area}_G(S \cap B(x, \varepsilon)) \ge c\varepsilon^2$$

whenever $x \in S$ and $S \cap B(x, \varepsilon)$ is a compact surface with its boundary contained in $\partial B(x, \varepsilon)$.

Area_G denotes the area with respect to the metric G.

The compacity assumption for M means that all the constants which are mentioned do not depend on the considered point x.

Proof of Theorem 2.10. We show that it is absurd if there is a sequence (z_k) in Δ^* converging to 0 such that $f_k(z_k) \to q' \neq q$.

(i) Assume that $|w_k| < |z_k|$ by taking a subsequence and relabeling.

Let $\rho_k(t) = w_k e^{it}$ for $t \in [0, 2\pi]$.

We claim that

$$f_k(\rho_k) \to q. \tag{1}$$

In fact for every $\alpha_k \in \rho_k$, we have

$$\liminf_{k\to\infty} d^J_{\overline{M}}(f_k(\alpha_k), q) \le \liminf_{k\to\infty} d^J_M(f_k(\alpha_k), f_k(w_k)) \le \liminf_{k\to\infty} d_{\Delta^*}(\alpha_k, w_k).$$

Since $d_{\Delta^*}(\alpha_k, w_k) = O(1/\log|w_k|)$ and $q \notin S^J_M(N)$, we have $f_k(\alpha_k) \to q$.

Let G be a length function on N. By Corollary 2.9 and Proposition 2.5 there exist relatively compact local coordinate neighborhoods U, W of q such that $\overline{U} \subset W$, U is diffeomorphic to the unit ball $B(q, 1)$ in some $\mathbb{C}^n$ centered at q and a positive constant c such that:

$$W \cap S^J_M(N) = \emptyset \quad \text{and} \quad q' \notin W. \tag{2}$$

$$K^J_M \ge c\,.\,G \quad \text{on} \quad W \cap M. \tag{3}$$

Since $f_k(\rho_k) \to q$ and $f_k(z_k) \to q'$, for sufficiently large k we have $f_k(\rho_k) \subset U$ and $f_k(z_k) \notin W$. Hence, there exists $z'_k \in \Delta^*$ such that $|w_k| < |z'_k| < |z_k|$ and $f_k(z'_k) \in \partial U$. By taking a subsequence if necessary, we may assume that $f_k(z'_k) \to p \in \partial U$. It follows from the equation (2) that $p \notin S^J_M(N)$.

Let $\mathcal{R}_k$ be the largest open annulus containing ρ_k and

$$f_k(\mathcal{R}_k) \subset U. \tag{4}$$

Since $f_k(z'_k) \to p \in \partial U$, then there exist $a_k \ge 0$ and $b_k < |z'_k|$ such that

$$\mathcal{R}_k = \{z \in \mathbb{C},\ a_k < |z| < b_k\}.$$

Let $\tilde{\mathcal{R}}_k = \{z \in \mathbb{C},\ |w_k| < |z| < |z'_k|\}$ and $\sigma_k = \{z \in \Delta^*,\ |z| = |z'_k|\}$. We have $f_k(\rho_k) \to q$ and in the same way as in the claim $f_k(\sigma_k) \to p \in \partial U$. Then for k sufficiently large, we have

$$f_k(\rho_k) \subset B\left(q, \frac{1}{4}\right),$$

$$f_k(\sigma_k) \subset W \setminus \overline{B}\left(q, \frac{3}{4}\right).$$

Therefore, there are points $c_k \in \tilde{\mathcal{R}}_k$, such that

$$f_k(c_k) \in \partial B\left(q, \frac{1}{2}\right).$$

By the Gromov's monotonicity lemma, there exist positive constants ε_0 and α such that for $\varepsilon \in]0, \inf(\varepsilon_0, 1/4)[$, we have

$$\mathrm{Area}_G(f_k(\tilde{\mathcal{R}}_k)) \geq \mathrm{Area}_G(f_k(\tilde{\mathcal{R}}_k) \cap B(f_k(c_k), \varepsilon)) \geq \alpha\varepsilon^2.$$

On the other hand, we denote by $\mathrm{Area}_{\Delta^*}(\tilde{\mathcal{R}}_k)$ the area of $\tilde{\mathcal{R}}_k$, with respect to the Poincaré metric on Δ^*. Then we have

$$\mathrm{Area}_{\Delta^*}(\tilde{\mathcal{R}}_k) = 2\pi\left(\frac{1}{\log(|z'_k|)} - \frac{1}{\log(|w_k|)}\right) \to 0.$$

It follows from the equations (3) and (4) that

$$\mathrm{Area}_G(f_k(\tilde{\mathcal{R}}_k)) \leq \frac{1}{c}\,.\,\mathrm{Area}_{\Delta^*}(\tilde{\mathcal{R}}_k) \to 0$$

and we get a contradiction. This proves the theorem in case (i).

(ii) Assume that $|z_k| < |w_k|$ by taking a subsequence and relabeling. As in the case (i), there exists a sequence (z'_k) in Δ^* such that $|z_k| < |z'_k| < |w_k|$ and $f_k(z'_k) \to p \in \partial U$. By considering the annulus $\tilde{\mathcal{R}}_k = \{z \in \mathbb{C}, |z'_k| < |z| < |w_k|\}$, we can reduce to the case (i). □

Corollary 2.13. *Let (M, J) be a relatively compact almost complex submanifold in an almost complex manifold (N, J) and let $f\colon \Delta^* \to (M, J)$ be a pseudoholomorphic curve. If there is a sequence (z_k) in Δ^* converging to 0 such that $f(z_k)$ converges to $q \notin S^J_M(N)$. Then f can be extended to a pseudoholomorphic curve $\tilde{f}\colon \Delta \to (N, J)$.*

Proof. Let $f\colon \Delta^* \to M$ be a pseudo-holomorphic map. By Theorem 2.10, f extends continuously from Δ to N and it is known that if f is continuous everywhere, differentiable and pseudo-holomorphic except on a discrete subset, then it's differentiable and thus pseudo-holomorphic in Δ, see [28] p. 169. □

Corollary 2.14. *Let (M, J) be a relatively compact hyperbolic almost complex submanifold in an almost complex manifold (N, J) and let $f\colon \Delta^* \to (M, J)$ be a pseudoholomorphic curve. If there is a sequence (z_k) in Δ^* converging to 0 such that $f(z_k)$ converges to $q \in M$. Then f can be extended to a pseudoholomorphic curve $\tilde{f}\colon \Delta \to (M, J)$.*

This is a direct consequence of Corollary 2.13, since if M is hyperbolic then every point of M is not a degeneracy point.

Corollary 2.15. *Let $f_k\colon \Delta^* \to M$ be a sequence of holomorphic curves. Assume that each f_k can be extended to holomorphic curve $\tilde{f}_k\colon \Delta \to N$. If there is a sequence (z_k) in Δ^* converging to 0 such that the sequence $(f_k(z_k))$ converges to $p \notin S_M^J(N)$, then $\tilde{f}_k(0)$ converges to p.*

Proof. If $\tilde{f}_k(0) \not\to q$ then by compactness, we may assume that $\tilde{f}_k(0) \to p \neq q$ since each $\tilde{f}_k$ is continuous there is a sequence (z_k) in Δ^* such that $z_k \to 0$ and $f_k(z_k) \to p$ which contradicts Theorem 2.10. □

The following theorem will be useful in the sequel.

Theorem 2.16. *Let (M, J) be a relatively compact almost complex submanifold hyperbolically embedded in an almost complex manifold (N, J). Then $\mathcal{O}_J(\Delta^*, M)$ is relatively compact in $\mathcal{O}_J(\Delta, N)$.*

We can consider $\mathcal{O}_J(\Delta^*, M)$ as a subspace of $\mathcal{O}_J(\Delta, N)$ as a consequence of Corollary 2.13 applied to the situation $S_M^J(N) = \emptyset$, that is (M, J) is hyperbolically embedded in (N, J).

Proof. Suppose that $\mathcal{O}_J(\Delta^*, M)$ is not relatively compact in $\mathcal{O}_J(\Delta, N)$, then by Ascoli, $\mathcal{O}_J(\Delta^*, M)$ is not equicontinuous at 0 and $p \in N$ that is, there exist an open neighborhood U of p, (z_n) a sequence in Δ^* and a sequence (f_n) in $\mathcal{O}_J(\Delta^*, M)$ such that $\tilde{f}_n(0)$ converges to p and $f_n(z_n) \notin U$, for each n. By compactness, we may suppose that the sequence $(f_n(z_n))$ converges to $q \notin U$, then by Corollary 2.15, we conclude the $(\tilde{f}_n(0))$ converges to q and we get a contradiction. □

2.3. Extension in higher dimensional manifolds. In this section, we are interested in the study of the extension of pseudoholomorphic maps into hyperbolically embedded almost complex manifolds.

Theorem 2.17. *Let C be a smooth pseudoholomorphic curve in an almost complex manifold (S, J') of real dimension 4 and let (M, J) be a relatively compact almost complex submanifold hyperbolically embedded in an almost complex manifold (N, J). Then every pseudoholomorphic map $f\colon (S \setminus C, J') \to (M, J)$ extends to a (J', J)-holomorphic map from S to N.*

The proof is based on the following lemma due to Joo [10].

Lemma 2.18. *Let A be a thin subset of an almost complex manifold X. Let (M, J) be a relatively compact almost complex submanifold in an almost complex manifold (N, J). Let $f\colon X \setminus A \to (M, J)$ be a holomorphic map. If f extends continuously to $\tilde{f}\colon X \to N$, then $\tilde{f}$ is a pseudoholomorphic map.*

As in [10], a closed subset A of X is called a *thin* subset if there exists a local foliation h of X by pseudo-holomorphic discs around p for every $p \in A$, which satisfies the following properties:
1. There is a positive constant $r < 1$ such that, $A_{z'} = \{w \in \Delta\colon h(z', w) \in C\}$ is a finite point set contained in the disc Δ_r for every $z' \in \Delta^{n-1}$.
2. There exist sequences (r_j) and (s_j) of real numbers less than 1 such that $r_j \to 0$ and the cylinders $\{(z', w)\colon |w| = r_j,\ |z'| < s_j\}$ do not intersect $h^{-1}(A)$ for every $j \in \mathbb{N}$.

EXAMPLE 2.19. Every smooth hypersurface when it exists in an almost complex manifold is thin. Especially, every smooth immersed pseudo-holomorphic curve in an almost complex manifold of real dimension 4 is thin.

Proof of Theorem 2.17. For an arbitrary point $p \in C$, choose a local foliation $h\colon \Delta \times \Delta \to X$ satisfying the conditions:
(a) h is a diffeomorphic onto a neighborhood of p and $h(0, 0) = p$.
(b) $h(\,\cdot\,, z')\colon \Delta \to X$ is a pseudoholomorphic embedding for every $z' \in \Delta$.
(c) For every $z' \in \Delta$, we have $\{w \in \Delta\colon h(w, z') \in C\} = \{0\}$.
We denote by f_w the map $f \circ h(\,\cdot\,, w)$ for $w \in \Delta$. Since for each $w \in \Delta$,

$$f_w\colon \Delta^* \to (M, J)$$

is a pseudoholomorphic curve defined on the punctured unit disc Δ^*, it can be extended to a pseudoholomorphic curve from the whole unit disc to N. We denote $\tilde{f}_w\colon \Delta \to (N, J)$ the extended map. Let (w_k) be a sequence in Δ which converges to $w_0 \in \Delta$.

We have only to prove that $\tilde{f}_{w_k}$ converges uniformly to $\tilde{f}_{w_0}$ in a neighborhood of 0.

Since (M, J) is hyperbolically embedded in (N, J), by Theorem 2.16, we may assume by considering a subsequence if necessary, that the sequence of pseudoholomorphic curves $(\tilde{f}_{w_k})$ converges uniformly on compact subsets of Δ to a pseudoholomorphic curve $\varphi\colon \Delta \to (N, J)$. Then it follows by the condition (c) that

$$\varphi(z) = \lim f_{w_k}(z) = f_{w_0}(z) = \tilde{f}_{w_0}(z) \quad \text{for} \quad z \in \Delta^*.$$

Therefore, it follows that $\tilde{f}_{w_0}$ coincides with φ on the unit disc and that $(\tilde{f}_{w_k})$ converges uniformly on every compact of Δ. This implies that $f \circ h$ is continuous in a neighborhood of $(0, w_0)$ and f can be extended continuously over X, denoted by $\tilde{f}$. By Lemma 2.18, we conclude that $\tilde{f}$ is a pseudoholomorphic map. □

3. Noguchi-type extension-convergence theorems

We prove a convergence-extension theorem for pseudoholomorphic curves proved in the complex case by Joseph-Kwack [9], which is analogous to the holomorphic extension theorem proved by Noguchi [25].

Theorem 3.1. *Let (M, J) be a relatively compact almost complex submanifold in an almost complex manifold (N, J). Let*

$$f_n : \Delta^* \to (M, J)$$

and

$$f : \Delta^* \to (M, J)$$

be a sequence of pseudoholomorphic curves and a pseudoholomorphic curve respectively. We suppose that there exists a sequence (z_n) in Δ^* converging to 0 such that the sequence $(f_n(z_n))$ converges to $p \notin S^J_M(N)$. Then f and f_n extends to Δ and if the sequence (f_n) converges to f uniformly on compact subsets of Δ^*, then the sequence $(\tilde{f}_n)$ converges to $\tilde{f}$ uniformly on compact subsets of Δ, where $(\tilde{f}_n)$ and $\tilde{f}$ are the extension to Δ of f_n and f respectively.

Proof. Since $p \notin S^J_M(N)$, then there exists a relatively compact neighborhood W of p such that $W \cap S^J_M(N) = \emptyset$. We claim the following:

Claim. *For every neighborhood V of p contained in W, there exist $r > 0$ and $n_0 \in \mathbb{N}$, such that*

$$f_n(\Delta^*_r) \subset V \cap M \quad \textit{for every} \quad n \geq n_0. \tag{5}$$

Otherwise, for each positive integer k there is some $w_k \in \Delta^*$ and an integer n_k such that $|w_k| < 1/k$ and $f_{n_k}(w_k) \notin V$. By compactness, we may assume that the sequence $(f_{n_k}(w_k))$ converges to q different of p. This contradicts Theorem 2.10.

To prove that f_n extends to a J-holomorphic curve in Δ, it is sufficient to verify that f_n has a finite energy. By Proposition 2.5, p is a J-hyperbolic point for M, that is, there exists a neighborhood U of p contained in W such that

$$K^J_M \geq c\,.\,G \quad \text{on} \quad U \cap M,$$

where c is a positive constant and G is a length function on N. By the claim, there exist $r > 0$ and $n_0 \in \mathbb{N}$, such that $f_n(\Delta^*_r) \subset U \cap M$. We conclude that

$$|f_n'(z)|_G \leq \frac{1}{c} K_{\Delta^*}(z), \quad \text{for every} \quad z \in \Delta^*_r.$$

Hence,

$$E(f_{n|\Delta^*_r}) = \frac{1}{2} \int_{\Delta^*_r} |f_n'(z)|^2_G \leq \frac{1}{2c^2} \int_{\Delta^*_r} K^2_{\Delta^*}(z) < \infty.$$

We denote by $\tilde{f}_n$ the extension of f_n to Δ. By the equation (5), we infer that $\tilde{f}_n(\Delta_r) \subset V \cap \overline{M}$ for every $n \geq n_0$. Choosing V sufficiently small, there exists a positive constant α such that $|f_n'(z)| \leq \alpha$ in Δ_r. By compactness, there exists a subsequence $(f_{\varphi(n)})$ which converges uniformly to J-holomorphic curve g. Since the sequence (f_n) converge to f uniformly on compact subsets of Δ^*, then g extends f.

Finally, we prove that the sequence $(\tilde{f}_n)$ converges uniformly to g in some neighborhood of 0. Otherwise, there exists a sequence (x_n) in Δ_r^* converging to 0 such that

$$|\tilde{f}_n(x_n) - g(x_n)|_{\text{euc}} \not\to 0,$$

where $|.|_{\text{euc}}$ denotes the Euclidean norm in $\mathbb{R}^m$. By Theorem 2.10, we have $\tilde{f}_n(x_n) \to p$ and $f(x_n) \to g(0) = p$ and we get a contradiction. $\square$

We prove a variant of Noguchi-type extension convergence theorems for pseudoholomorphic maps defined on $S \setminus C$ to hyperbolically embedded submanifold, where C is a smooth, pseudoholomorphic curve in an almost complex manifold S of real dimension 4.

Theorem 3.2. *Let C be a smooth pseudoholomorphic curve in an almost complex manifold (S, J') of real dimension 4 and let (M, J) be a relatively compact almost complex submanifold hyperbolically embedded in an almost complex manifold (N, J). Let*

$$f_n : (S \setminus C, J') \to (M, J)$$

and

$$f : (S \setminus C, J') \to (M, J)$$

be a sequence of (J', J)-holomorphic maps and (J', J)-holomorphic map respectively.

If the sequence (f_n) converges to f uniformly on compact subsets of $S \setminus C$, then the sequence $(\tilde{f}_n)$ converges to $\tilde{f}$ uniformly on compact subsets of S, where $\tilde{f}_n$ and $\tilde{f}$ are the (J', J)-holomorphic extensions to S of f_n and f respectively.

Proof. By Theorem 2.17, each f_n and f extends to (J', J)-holomorphic maps $\tilde{f}_n : X \to N$ and $\tilde{f} : X \to N$ respectively. The question of convergence arises in the neighborhood of a point $p \in C$. Choose a local foliation $h : \Delta \times \Delta \to X$ satisfying the following conditions:

(a) h is a diffeomorphic onto a neighborhood of p and $h(0, 0) = p$.

(b) $h(\cdot, z') : \Delta \to X$ is a pseudoholomorphic embedding for every $z' \in \Delta$.

(c) For every $z' \in \Delta$, we have $\{w \in \Delta : h(w, z') \in C\} = \{0\}$.

We will denote by

$$\varphi_n := f_n \circ h \colon \Delta^* \times \Delta \to (M, J)$$

and

$$\tilde{\varphi}_n := \tilde{f}_n \circ h \colon \Delta \times \Delta \to (M, J)$$

and for every $r \in]0, 1[$, $S_r := \{z \in \Delta;\ |z| = r\}$. Let $\tilde{f} \circ h(0) = p$. Suppose that the sequence $(\tilde{\varphi}_n)$ does not converge uniformly on some neighborhood of 0, then we can pick a relatively compact neighborhood W of p diffeomorphic to $B(1)$ the unit ball in some $\mathbb{C}^m$, such that for each positive integer k and $r \in]0, 1[$ there are infinitely many n such that

$$\tilde{\varphi}_n(\Delta_{1/k} \times \Delta_r) \not\subset W.$$

There is some k_0 and r_0 such that

$$\tilde{\varphi}(\overline{\Delta_{1/k_0}} \times \Delta_r) \subset B\left(\frac{1}{8}\right).$$

Since the sequence (φ_n) converge uniformly on $S_{1/k} \times \overline{\Delta_{r_0}}$, then there exists a subsequence (φ_{n_k}) of (φ_n) and a sequence (z'_k) in Δ_{r_0} converging to 0 such that

$$\varphi_{n_k}(S_{1/k}, z'_k) \subset B\left(\frac{1}{4}\right)$$

and

$$\tilde{\varphi}_{n_k}(\Delta_{1/k}, z'_k) \not\subset B(1).$$

Hence, for each $k \geq k_0$, there is a point $z_k \in \Delta_{1/k}$ such that

$$g_{n_k}(z_k) := \tilde{\varphi}_{n_k}(z_k, z'_k) \in S_{1/2},$$

where g_{n_k} is the pseudoholomorphic curve defined by $g_{n_k} = \tilde{\varphi}_{n_k}(\,\cdot\,, z'_k)$.

By the Gromov's monotonicity lemma, there exist positive constants ε_0 and α such that for $\varepsilon \in]0, \inf(\varepsilon_0, 1/8)[$, we have

$$\mathrm{Area}_G(g_{n_k}(\Delta_{1/k})) \geq \mathrm{Area}_G(g_{n_k}(\Delta_{1/k}) \cap B(g_{n_k}(z_k), \varepsilon)) \geq \alpha\varepsilon^2,$$

where G is a length function in N.

On the other hand, since (M, J) is hyperbolically embedded in (N, J) then, there exists a positive constant c such that

$$K_M^J \geq c \,.\, G.$$

The restriction of g_{n_k} in Δ^* is a pseudoholomorphic curve then

$$g_{n_k}^*(G) \leq \frac{1}{c} \,.\, g_{n_k}^*(K_M^J) \leq \frac{1}{c} K_{\Delta^*}.$$

As a consequence, we have

$$\mathrm{Area}_G(g_{n_k}(\Delta_{1/k})) = \mathrm{Area}_G(g_{n_k}(\Delta_{1/k}^*)) \leq \frac{1}{c} \,.\, \mathrm{Area}_{\Delta^*}(\Delta_{1/k}^*) \to 0.$$

We get a contradiction. □

When M is compact hyperbolic, we can strengthen the last theorem in the following form.

Theorem 3.3. *Let A be a thin subset in an almost complex manifold (X, J'), (M, J) be a compact hyperbolic almost complex manifold and (J_n) be a sequence of almost complex structures on M converging to J for the $\mathcal{C}^\infty$-topology.*

Let

$$f_n : (X \setminus A, J') \to (M, J_n)$$

and

$$f : (X \setminus A, J') \to (M, J)$$

be a sequence of pseudo-holomorphic maps and a pseudo-holomorphic map respectively.

If (f_n) converges to f uniformly on compact subsets of $X \setminus A$, then $(\tilde{f}_n)$ converges to $\tilde{f}$ uniformly on compact subsets of X, where $\tilde{f}_n : (X, J') \to (M, J_n)$ and $\tilde{f} : (X, J') \to (M, J)$ are the extensions to X of f_n and f respectively.

The proof of the theorem is based on the following two lemmas.

Lemma 3.4. *Let (M, J) be a compact hyperbolic almost complex manifold and G be a length function on M. Then there exists an open neighborhood $\mathcal{U}$ of J and a positive constant c such that*

$$K_M^{J'} \geq c \,.\, G \quad \textit{for every} \quad J' \in \mathcal{U}.$$

Proof. We assume that there is no such c and neighborhood $\mathcal{U}$. Then there is a sequence of tangent vectors (ξ_k) in TM and a sequence of almost complex structures (J_k) converging to J such that $|\xi_k|_G = 1$ and $K_M^{J_k}(\xi_k) \to 0$. By considering a subsequence, we may assume that the sequence $(K_M^{J_k}(\xi_k))$ is monotone decreasing. Hence, there is a monotone increasing sequence (r_k) of positive numbers tending to $+\infty$ and a family of pseudoholomorphic curves $f_k\colon \Delta_{r_k} \to (M, J_k)$ such that $f_k'(0) = \xi_k$. By applying Brody's reparametrization theorem [4] to each f_k which remains valid in the almost complex case, we obtain a sequence of pseudoholomorphic curves $\varphi_k\colon \Delta_{r_k} \to (M, J_k)$ such that: $|\varphi_k'(z)|_G \le r_k^2/(r_k^2 - |z|^2)$ on Δ_{r_k} and the equality holds at the origin 0. By the compactness theorem [22], we can extract a subsequence of (φ_k) which converges uniformly with all derivatives on compact sets to a pseudoholomorphic curve $\varphi\colon \mathbb{C} \to (M, J)$. The mapping φ is non constant since $|\varphi'(0)|_G = \lim|\varphi_k'(0)|_G = 1$ and $|\varphi'(z)|_G \le 1$ for each $z \in \mathbb{C}$. This contradicts the hyperbolicity of M. □

The following lemma is due to Gaussier-Sukhov [7].

Lemma 3.5. *Let (M, J) (resp. (M', J')) be a smooth almost complex manifold. Let (J_n) (resp. (J_n')) be a sequence of almost complex structures on M (resp. M') converging in the $\mathcal{C}^\infty(M)$-topology (resp. $\mathcal{C}^\infty(M')$-topology) to J (resp. J'). For every n, let $f_n \in \mathcal{O}_{(J_n', J_n)}(M', M)$. Assume that (f_n) converges uniformly on compact subsets of M to a map f. Then f belongs to $\mathcal{O}_{(J', J)}(M', M)$.*

It is well-known from the standard elliptic estimates for the Cauchy-Green kernel (see Sikorav [28]) showing that the limit in the compact open topology of a sequence of J-holomorphic discs also is a J-holomorphic curve.

Proof of Theorem 3.3. It is known, that every holomorphic map $g\colon X \setminus A \to M$ extends to a holomorphic map $\tilde{g}\colon X \to M$, see [10]. Hence, each f_n extends to a pseudoholomorphic map $\tilde{f}_n\colon (X, J') \to (M, J_n)$.

Let G be a length function on M. By Lemma 3.4, there exist a positive constant c and an integer n_0 such that $K_M^{J_n} \ge c \,.\, G$ for each $n \ge n_0$, hence

$$d_{c.G}(\tilde{f}_n(z), \tilde{f}_n(w)) \le d_M^{J_n}(\tilde{f}_n(z), \tilde{f}_n(w)) \quad \text{for every} \quad z, w \in X.$$

Let $(\tilde{f}_{\varphi(n)})$ be an arbitrary subsequence of $(\tilde{f}_n)$. Since $\tilde{f}_{\varphi(n)}$ is a $(J', J_{\varphi(n)})$-pseudoholomorphic map from X to M, we have for $n \ge n_0$

$$d_{c.G}(\tilde{f}_{\varphi(n)}(z), \tilde{f}_{\varphi(n)}(w)) \le d_X^{J'}(z, w) \quad \text{for every} \quad z, w \in X.$$

By Ascoli, we infer that the family $(\tilde{f}_{\varphi(n)})$ is equicontinuous and we can extract a subsequence $\tilde{f}_{\varphi\circ\psi(n)}$ which converges to a map g. By Lemma 3.5, we conclude that the map g is (J', J)-holomorphic and coincides with f on $X \setminus A$. Hence, $\tilde{f} = g$ and we conclude finally that the sequence $(\tilde{f}_n)$ converges uniformly on each compact subset of X to $\tilde{f}$. □

In particular, if A is an immersed curve in an almost complex surface, we get

Corollary 3.6. *Let C be a smooth pseudoholomorphic curve in an almost complex manifold (S, J') of real dimension 4 and let (M, J) be a compact hyperbolic almost complex manifold and (J_n) be a sequence of almost complex structures on M converging to J for the $\mathcal{C}^\infty$-topology.*

Let

$$f_n : (S \setminus C, J') \to (M, J_n)$$

and

$$f : (S \setminus C, J') \to (M, J)$$

be a sequence of pseudo-holomorphic maps and a pseudo-holomorphic map respectively.

If (f_n) converges to f uniformly on compact subsets of $S \setminus C$, then $(\tilde{f}_n)$ converges to $\tilde{f}$ uniformly on compact subsets of S, where $\tilde{f}_n : (S, J') \to (M, J_n)$ and $\tilde{f} : (S, J') \to (M, J)$ are the extension to S of f_n and f respectively.

Now, we prove another variant of Noguchi extension convergence theorem for J-holomorphic curves.

Theorem 3.7. *Let (M, J) be a relatively compact hyperbolic almost complex submanifold in an almost complex manifold (N, J). Assume that there is a neighborhood U of ∂M, the boundary of M in N such that $U \cap M$ is hyperconvex. Then, each pseudoholomorphic curve $f : \Delta^* \to M$ extends to a pseudoholomorphic curve $\tilde{f} : \Delta \to M$.*

Moreover, if $(f_n : \Delta^* \to (M, J))$ is a sequence of pseudoholomorphic curves that converges uniformly on compact subsets of Δ^* to a pseudoholomorphic curve $f : \Delta^* \to (M, J)$, then the sequence $(\tilde{f}_n)$ converges uniformly on compact subsets of Δ to $\tilde{f}$, where $\tilde{f}_n$ and $\tilde{f}$ are the extension to Δ of f_n and f respectively.

Proof. It suffices to prove by Corollary 2.14, that there is a sequence (z_n) in Δ^* converging to 0 such that the sequence $(f(z_n))$ converges to a point of M. Suppose that this is not the case. Then there exists $r \in]0, 1[$ such that $f(\Delta_r^*) \subset U$. Let φ be a plurisubharmonic function of $U \cap M$. Then the function $g = \varphi \circ f$ is subharmonic on Δ_r^*. By assumption g extends continuously to a function $\tilde{g}$ which remains subharmonic on Δ_r. We have $\tilde{g}(z) < 0$ for every $z \in \Delta_r^*$ and $\tilde{g}(0) = 0$, so $\tilde{g}$ attains its maximum at the origin. We get a contradiction by the maximum principle.

For the second assertion, let $(\tilde{f}_{\varphi(n)})$ be an arbitrary subsequence of $(\tilde{f}_n)$. Since $\tilde{f}_{\varphi(n)}$ is a pseudoholomorphic curve from Δ to M, we have

$$d_M^J(\tilde{f}_{\varphi(n)}(z), \tilde{f}_{\varphi(n)}(w)) \leq d_\Delta(z, w) \quad \text{for every} \quad z, w \in \Delta.$$

Hence, the family $(\tilde{f}_{\varphi(n)})$ is equicontinuous and by Ascoli, we can extract a subsequence $\tilde{f}_{\varphi \circ \psi(n)}$ which converges to a pseudoholomorphic map g. But the map g coincides with f on Δ^*, hence $\tilde{f} = g$ and we conclude finally that the sequence $(\tilde{f}_n)$ converges to $\tilde{f}$ uniformly on each compact subset of Δ. $\square$

4. Relative intrinsic pseudodistance

Let (M, J) be a relatively compact almost complex submanifold in an almost complex manifold (N, J). As in Kobayashi [15], we shall introduce a relative pseudodistance $d^J_{M,N}$ on $\overline{M}$ and we prove that M is hyperbolically embedded in N, if and only if, $d^J_{M,N}$ is a distance on $\overline{M}$.

Let $\mathcal{F}^J_{M,N} \subset \mathcal{O}_J(\Delta, N)$ be the family of pseudoholomorphic curves such that $f^{-1}(N \setminus M)$ is either empty or a singleton. Thus, each element of $\mathcal{F}^J_{M,N}$ maps all Δ, with the exception of possibly one point, into M. The exceptional point is of course mapped into $\overline{M}$.

We define a pseudodistance $d^J_{M,N}$ on $\overline{M}$ in the same way as d^J_N, but using only chains of pseudoholomorphic curves belonging to $\mathcal{F}^J_{M,N}$. Namely, writing $l(\alpha)$ for the length of a chain α of pseudoholomorphic curves, we set

$$d^J_{M,N}(p, q) = \inf_{\alpha} l(\alpha),$$

where the infimum is taken over all chains α of pseudoholomorphic curves from p to q which belong to $\mathcal{F}^J_{M,N}$.

The interesting case is where M is the complement of a hypersurface in N, but since a hypersurface does not exist even locally, we will consider the case where $N := S$ is a compact almost complex manifold of real dimension 4 and $M = S \setminus C$ where C is a pseudoholomorphic curve immersed in S. In this case, any pair of points p, q in $\overline{S \setminus C} = S$ can be joined by a chain of pseudoholomorphic curves belonging to $\mathcal{F}^J_{S\setminus C,S}$, so that $d^J_{S\setminus C,S}(p, q) < \infty$ for $p, q \in S$.

Since

$$\mathcal{O}_J(\Delta, S \setminus C) \subset \mathcal{F}^J_{S\setminus C,S} \subset \mathcal{O}_J(\Delta, S),$$

we have

$$d^J_S \le d^J_{S\setminus C,S} \le d^J_{S\setminus C}.$$

Let (C', J') be a pseudoholomorphic curve in an almost complex manifold (S', J') of real dimension 4. If $f : (S, J) \to (S', J')$ is a (J, J')-holomorphic map such that $f(S \setminus C) \subset S' \setminus C'$, then

$$d^{J'}_{S'\setminus C',S'}(f(p), f(q)) \le d^J_{S\setminus C,S}(p, q), \quad p, q \in S.$$

Proposition 4.1. *Let (C, J) be a pseudoholomorphic curve in an almost complex manifold (S, J) of real dimension 4. Then the pseudodistance $d^J_{S\setminus C,S}$ is continuous on $S \times S$, moreover if it is a distance on S, then it induces the standard topology on S.*

The proof is similar to Kruglikov-Overholt [19]. Our principal result in this section is the following theorem.

Theorem 4.2. *Let (C, J) be a pseudoholomorphic curve in a compact almost complex manifold (S, J) of dimension* 4*. The following conditions are equivalent*:
(a) $d^J_{S\setminus C,S}$ *is a distance on* S.
(b) $(S \setminus C, J)$ *is hyperbolically embedded in* (S, J).

Proof. (a) $\Rightarrow$ (b) Let (p_n) and (q_n) be sequences in $S \setminus C$ with $p_n \to p \in S$ and $q_n \to q \in S$. If $d^J_{S\setminus C}(p_n, q_n) \to 0$ then $d^J_{S\setminus C,S}(p_n, q_n) \to 0$. This implies that $p = q$.

(b) $\Rightarrow$ (a) By theorem of Sikorav the topology $\mathcal{C}^0$ in the space of pseudoholomorphic curves coincides with the topology of the uniform convergence with all derivatives in the compact sets, the proof of Kobayashi [15] in the complex case is still valid. □

5. The automorphism group

Kobayashi [16] proved that the automorphism group of a compact hyperbolic complex manifold is finite, later Miyano-Noguchi [23] showed that the automorphism group of $X \setminus D$ is finite where D is a normal crossing divisor in a compact complex manifold X and $X \setminus D$ is hyperbolically embedded in X. Recently Kruglikov-Overholt [19] and Kobayashi [17] proved that the automorphism group of a compact hyperbolic almost complex manifold is finite.

The main result of this section is the following

Theorem 5.1. *Let (C, J) be a pseudoholomorphic curve in a compact almost complex manifold (S, J) of real dimension* 4*, such that $(S \setminus C, J)$ is hyperbolically embedded in (S, J). Then the automorphism group $\mathrm{Aut}_J(S \setminus C)$ of $S \setminus C$ is finite.*

For the proof, we will need the following lemma due to van Dantzig and van der Waerden [5]. For its proof, see also Kobayashi-Nomizu [13] pp. 46–50.

Lemma 5.2. *The group $I(X)$ of isometries of a connected, locally compact metric space X is locally compact with respect to the compact-open topology, and for any point $x \in X$ and any compact subset $K \subset X$, the subset $\{f \in I(X);\ f(x) \in K\}$ is compact. In particular, at any point $x \in X$ the isotropy subgroup $I_x(X)$ is compact. If X is moreover compact, then $I(X)$ is compact.*

Proof of Thereom 5.1. First, we shall prove that $\mathrm{Aut}_J(S\setminus C)$ is compact endowed with the compact-open topology on $S \setminus C$. We set

$$\mathrm{Aut}_J(S, S \setminus C) := \{f \in \mathrm{Aut}_J(S);\ f(S \setminus C) \subset S \setminus C\}$$

equipped with the compact-open topology on S. By Theorem 3.2, the restriction map $\mathrm{Aut}_J(S, S \setminus C) \to \mathrm{Aut}_J(S \setminus C)$ is a homeomorphism.

Since $S \setminus C$ is hyperbolically embedded in S then by Theorem 5, $d^J_{S\setminus C,S}$ is a distance which induces the standard topology of S. Let $I(S, S \setminus C)$ be the group of isometries of S with respect to the relative distance $d^J_{S\setminus C,S}$, it follows from Lemma 5.2 that $I(S, S \setminus C)$ is compact. It remains to prove that $\mathrm{Aut}_J(S, S \setminus C)$ is closed in $I(S, S \setminus C)$.

Let (f_j) be a sequence in $\mathrm{Aut}_J(S, S \setminus C)$ which converges to $f \in I(S, S \setminus C)$. By Lemma 3.5, f is a (J, J)-holomorphic map from S to S. Since $f_j \in \mathrm{Aut}_J(S, S \setminus C)$, then there exists a sequence (g_j) in $\mathrm{Aut}_J(S, S \setminus C)$, such that $f_j \circ g_j = g_j \circ f_j = \mathrm{Id}_S$. By considering a subsequence and by applying again Lemma 3.5, we suppose that (g_j) converges uniformly on compact subsets to (J, J)-holomorphic map g from S to S such $f \circ g = g \circ f = \mathrm{Id}$. Hence, $f \in \mathrm{Aut}_J(S)$.

We infer that $f(S \setminus C) \subset S \setminus C$. Let $p_0 \in S \setminus C$ such that $q_0 = f(p_0) \in S \setminus C$. For any $p \in S \setminus C$, we have

$$d^J_{S\setminus C}(f_j(p), f_j(p_0)) = d^J_{S\setminus C}(p, p_0).$$

Then for some integer j_0, we have for all $j \geq j_0$

$$f_j(p) \in B'_{d^J_{S\setminus C}}(q_0, d^J_{S\setminus C}(p, p_0) + 1).$$

$S \setminus C$ is complete hyperbolic since it is locally complete hyperbolic [6] and hyperbolically embedded, see [12]. Therefore the closed ball $B'_{d^J_{S\setminus C}}(q_0, d^J_{S\setminus C}(p, p_0) + 1)$ is compact hence, $f(p) = \lim f_j(p)$ in $S \setminus C$. We conclude finally that $f \in \mathrm{Aut}_J(S, S \setminus C)$.

By a theorem of Bochner-Montgomery [3], a locally compact group of differentiable transformations of manifold is a Lie transformation group. Hence, $\mathrm{Aut}_J(S \setminus C)$ is a Lie group. Kruglikov-Overholt [19] proved that no almost complex Lie group of positive dimension acts effectively as a pseudoholomorphic transformation group on a hyperbolic almost complex manifold. Hence the Lie algebra of $\mathrm{Aut}_J(S, S \setminus C)$ is trivial. □

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